WEDGE FLOWS OF A PLASTICOVISCOUS MEDIUM WITH NONLINEAR VELOCITY

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ABSTRACT: The properties of plasticoviscous media have been the subject of numerous studies, in particular [1–5]. This paper deals with the problem of plasticoviscous flow in the absence of a pressure drop of a medium with nonlinear viscosity in pure shear in a region wedge-shaped in plan, and with the problem of flow under the influence of a pressure drop, when one face of the wedge moves parallel to the edge.

1. We will consider the flow of an isotropic plasticoviscous medium with nonlinear viscosity between two infinitely long rigid cylinders (Fig. 1). The cross section of the contour of one of the cylinders S_1 is wedge-shaped, the cross section of the second cylinder S_2 takes the shape of a smooth closed curve asymptotically approaching the contour S_1 at infinity. The two cylinders have parallel generators. The first cylinder is fixed, the second moves at a constant velocity u_0 parallel to the generators.

Let the z axis be directed along the generators of the cylinders in the direction of motion of the second cylinder. We set up the x and y axes in the plane of the cross section of the first cylinder. The velocity u(x, y) of each particle of the medium is directed along the z axis.

The nonlinear relation between the shearing stress τ and the shear rate γ is taken in the form

$$\eta\gamma = (\tau - k)^{\mu} \qquad (\mu > 0), \tag{1.1}$$

where k is the yield point, and η, μ the coefficient of viscosity and the viscosity exponent. Retaining the previous notation, we go over to dimensionless quantities. We refer the velocity u(x, y) to the quantity u_0, the stress τ to the yield point k, and quantities with the dimension of length to the quantity k^{μ}/η. The Eq. (1.1) may be rewritten in the dimensionless form

$$\gamma = (\tau - 1)^{\mu}. \tag{1.2}$$

Following the ideas of [2], we go from the plane xy to the orthogonal network of coordinates u and v formed by the lines of equal velocity u = const and the lines of stresses v = const, to which the vector τ is tangential.

The equation for the function $u(\tau, \varphi)$ has the form

$$\frac{\gamma\tau}{\gamma'}\frac{\partial^2 u}{\partial\tau^2} + \frac{2\gamma - \tau\gamma'}{\gamma'}\frac{\partial u}{\partial\tau} + \frac{\partial^2 u}{\partial\varphi^2} = 0, \quad \gamma' = \frac{d\gamma}{d\tau}. \tag{1.3}$$

The boundary conditions are

$$u = 0 \text{ on } S_1, \qquad u = 1 \text{ on } S_2, \qquad u = 0 \text{ at } \tau = 1. \tag{1.4}$$

The solution of Eq. (1.3) with boundary conditions (1.4) will be found in the form

$$u = T(\tau)\cos\lambda\varphi. \tag{1.5}$$

Substituting (1.5) and (1.3) and satisfying the second of conditions (1.4), we obtain

$$u(\tau, \varphi) = A_n(\tau - 1)^{1+\mu}\Phi_n \cos\lambda_n\varphi, \tag{1.6}$$

where λ_n are the eigenvalues of the problem, and $\Phi_n(2 - \alpha_n, 2 - \beta_n, 2 + \mu, 1 - \tau)$ is a hypergeometric function,

$$\alpha_n = {}^1/_2(1 - \mu) + [{}^1/_4(1 - \mu)^2 + \mu\lambda_n^2]^{1/2},$$
$$\beta_n = {}^1/_2(1 - \mu) - [{}^1/_4(1 - \mu)^2 + \mu\lambda_n^2]^{1/2}. \tag{1.7}$$

We denote by ω the cone angle of the contour S_1. If we set

$$\lambda_n = \frac{(2n+1)\pi}{\pi - \omega}, \tag{1.8}$$

then the solution (1.6) will satisfy the first and second of boundary conditions (1.4). Summing the particular solutions (1.5), we obtain the general solution

$$u = \sum_{n=1}^{m} A_n(\tau - 1)^{1+\mu}\Phi_n\cos\lambda_n\varphi. \tag{1.9}$$

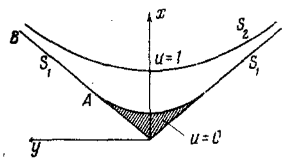

Fig. 1

We will use the third of conditions (1.4) as an equation for determining the moving contour S_2.

Evaluating the derivatives $\partial x/\partial\varphi$, $\partial x/\partial\tau$, $\partial y/\partial\varphi$ and $\partial y/\partial\tau$, we find

$$x = x(\tau, \varphi) = \sum_{n=1}^{m} A_n(K_n\cos\varphi\cos\lambda_n\varphi + L_n\sin\varphi\sin\lambda_n\varphi),$$
$$y = y(\tau, \varphi) = \sum_{n=1}^{m} A_n(K_n\sin\varphi\cos\lambda_n\varphi - L_n\cos\varphi\sin\lambda_n\varphi), \tag{1.10}$$

$$K_n = \frac{\mu(\mu+1)}{(1-\alpha_n)(1-\beta_n)}(1 - \Phi_n^*) - (1 - \tau)\Phi_n + \frac{\mu+1}{\lambda_n^2 - 1},$$
$$L_n = \lambda_n[K_n + (1 - \tau)\Phi_n]. \tag{1.11}$$

Here A_n is an arbitrary constant, and $\Phi_n^*(1 - \alpha_n, 1 - \beta_n, 1 + \mu, 1 - \tau)$ is a hypergeometric function. The equation for the contour of the rigid core $x_1(\varphi)$, $y_1(\varphi)$ is obtained from (1.10) by setting τ = 1 in those expressions

$$x_1(\varphi) = \sum_{n=1}^{m} A_nK_n(1)(\cos\varphi\cos\lambda_n\varphi + \lambda_n\sin\varphi\sin\lambda_n\varphi),$$
$$y_1(\varphi) = \sum_{n=1}^{m} A_nK_n(1)(\sin\varphi\cos\lambda_n\varphi - \lambda_n\cos\varphi\sin\lambda_n\varphi). \tag{1.12}$$

The stress vector τ is orthogonal to the contours S_1 and S_2; therefore on AB the angle $\varphi = -(\pi - \omega)/2$. If in (1.10) we set $\varphi = -(\pi - \omega)/2$, we obtain $y = x \operatorname{tg} \omega/2$, i.e., the equation of the line AB. Setting $\varphi = (\omega - \pi)/2$ and $\tau = 1$ in (1.10), we find the position of the point A at which the contour of the core and the line AB meet:

$$x = \sum_{n=1}^{m} \lambda_nA_nK_n(1)\cos\frac{\omega}{2}, \quad y = \sum_{n=1}^{m} \lambda_nA_nK_n(1)\sin\frac{\omega}{2}. \tag{1.13}$$

At $\varphi = 0$ from (1.12) we find the coordinates of the points of intersection of the contour of the core and the x axis:

$$x = \sum_{n=1}^{m} A_n\frac{\mu+1}{\lambda_n^2 - 1}. \tag{1.14}$$

Obviously, at $\varphi = -(\pi - \omega)/2$ from (1.12) we find the same values of the coordinates of the point A. To find the coordinates of the point

A from both (1.10) and (1.12) it is necessary to take the same value of the angle φ; therefore the contour of the core and the contour S_1 meet smoothly at the point A, having a common tangent. This can also be seen by calculating the slope of the tangent at the point A to the line (1.12):

$$\lambda_n = (2n+1),\ y_A = 0,\ x_A = \sum_{n=1}^{m} A_n \frac{\lambda_n(\mu+1)}{\lambda_n^2 - 1} \quad (\omega \to 0),$$
$$\lambda_n \to \infty,\quad x_A = z_A = 0 \qquad (\omega \to \pi).$$

As the angle ω increases to a straight angle, the stagnant zone diminishes to zero.

On the basis of (1.9) and (1.10) it is easy to see that at infinity all the lines u = const asymptotically approach the contour S_1.

2. We will now consider the flow of a plasticoviscous medium in a wedge under the action of a pressure drop $P(r, \varphi)$ when one face is fixed and the other moves at a constant velocity u_0 parallel to the edge.

We will assume that the flow inside the wedge is described by the function $u = u(\varphi)$. The shearing stress-shear relation is written in the form

$$\tau = k + F(\gamma). \tag{2.1}$$

From the equilibrium equation

$$\frac{1}{r}\frac{\partial \tau}{\partial \varphi} = P(r, \varphi) \tag{2.2}$$

and from (2.1) we find that in the case in question

$$P(r, \varphi) = \frac{u''}{r^2}\frac{dF}{d\gamma},\quad \gamma = \frac{1}{r}u',\quad \tau = k + F\left(\frac{1}{r}u'\right), \tag{2.3}$$

i.e., as the apex of the wedge is approached the values of the shear rate and shearing stress depend on the direction of approach.

We write the boundary conditions for the function u in the form

$$u = 0 \ \text{ at } \ \varphi = 0,\ u = u_0 \ \text{ at } \ \varphi = \omega. \tag{2.4}$$

At $P(r, \varphi)$ from (2.3) and (2.4) we obtain

$$u = \varphi u_0/\omega, \tag{2.5}$$

i.e., particles of the medium on a ray drawn from the apex of the wedge move at different velocities.

For the given $P(r, \varphi)$ from (2.3) and (2.4) we find $u = u(\varphi)$.

We calculate the force T applied to the part of the wedge face [0, r]:

$$T = \int_0^r \tau dr = kr - u' \int_\gamma^\infty \frac{F(\gamma)}{\gamma^2} d\gamma. \tag{2.6}$$

Since the force T is a finite quantity, integral (2.6) imposes a limitation on the choice of $F(\gamma)$. Thus, at points where γ is large, the relation $F(\gamma)$ must be such that the force T is a finite quantity.

If we take $F(\gamma) = \eta\gamma^m$, then from (2.6) it is easy to see that the inequality m = 1 must be satisfied, i.e., at points where γ increases without bound, the viscosity cannot be linear.

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